Coherence thresholds in models of language change and evolution: The effects of noise, dynamics, and network of interactions

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A simple model of language evolution proposed by Komarova, Niyogi, and Nowak is characterized by a payoff in communicative function and by an error in learning that measure the accuracy in language acquisition. The time scale for language change is generational, and the model's equations in the mean-field approximation are a particular case of the replicator-mutator equations of evolutionary dynamics. In well-mixed populations, this model exhibits a critical coherence threshold; i.e., a minimal accuracy in the learning process is required to maintain linguistic coherence. In this work, we analyze in detail the effects of different fitness-based dynamics driving linguistic coherence and of the network of interactions on the nature of the coherence threshold by performing numerical simulations and theoretical analyses of three different models of language change in finite populations with two types of structure: fully connected networks and regular random graphs. We find that although the threshold of the original replicator-mutator evolutionary model is robust with respect to the structure of the network of contacts, the coherence threshold of related fitness-driven models may be strongly affected by this feature.

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I. INTRODUCTION

Statistical physics has become a powerful framework to investigate the collective behavior of individuals and is playing an increasingly prominent role in quantitative social sciences studies. A case in point is opinion dynamics [1], which aims at describing the emergent social behavior by considering models with simple rules of opinion formation, through which "agents" update their internal state, or opinion, through the interactions with other "agents." The interactions are typically local rules that consist in (a) following the majority or (b) random neighbor imitation, two simple mechanisms that have been studied for decades as models for the dynamics of Ising spin systems, known in the physics literature as the Glauber and Voter models, respectively [2,3].

Traditional statistical physics models consider particles (spins, agents) interacting (i) with all the other particles as analytical solutions are often possible in this mean-field limit or (ii) with a number of neighbors located on the vertices of regular lattices in d dimensions, the topology characteristic of crystalline solids. Recently, however, the field of complex networks [4,5] paved the way for a better description of social dynamics by providing adequate models for networks of social interactions that are neither well mixed as in (i) nor completely regular as in (ii). Since then, numerous studies have considered the evolution of opinion models on complex networks and investigated the effects of the network topology on the model's dynamical behavior. In particular, novel, nontrivial behavior has been found for the ordering dynamics of the zero-temperature Glauber and Voter models on complex networks [6–11].

Language competition may be viewed as a particular case of consensus problems and as such has motivated related studies [12,13]. Other aspects of language dynamics include language change and evolution and language learning. In this context the pioneering work of [14,15] considers a biologically inspired evolutionary model where the errors in learning are assumed to be the major determinant for language change. This class of models for language change is based on the assumption that languages evolve like individuals in a population: the fittest survive and spread; the less fit are eliminated. The two driving forces of evolution, selection and mutation (i.e., language transmission with a bias that favors the fittest or the dominant language and errors in the transmission process), are incorporated into the replicator-mutator dynamics equations, and the time scale for change is generational.

In the framework of replicator-mutator evolutionary models for language dynamics, the question that arises is, how accurately do children have to learn the language of their parents in order for the population to maintain a coherent language? The question was answered in a series of papers [14,16,17] that show that in the strong selection limit a critical threshold, largely determined by the error rate of language acquisition, exists for infinite [14] and finite [16] well-mixed populations, irrespective of the number of languages in competition [14].

More recently, language games such as the naming game used to model the emergence of language understood as a consensual lexicon [18,19] have attracted the attention of the physics community [20,21]. This class of models focuses on the horizontal transmission and "creation" of language as a result of peer-to-peer interaction, in contrast with vertical transmission, the basic scheme of language change in models inspired by biological evolution. Apart from the time scale for change, which is no longer generational, here the question is to establish when the dynamics of a set of interacting agents that can choose among several options leads to consensus, or alternatively, when a state with several coexisting options, or language diversity, prevails.

Another class of evolutionary models of languages with analogies with the theories of population genetics was proposed by Baxter *et al.* [22]. The model was solved in the limit of a single speaker as well as for multiple speakers in the mean-field approximation, and (in these limits) it was shown to be related to the model of Abrams and Strogatz [23] for the extinction of languages.

Evolutionary age-structured models based on the Penna model [24] have also explored the similarities between biological evolution and language learning and competition. One model, applied to language competition, revealed the existence of a (first-order) phase transition between the dominance of a single language and language diversity, driven by random mutations [25], which is akin to the transitions described in this paper. In a different context, that of learning foreign languages, a similar noise-driven transition was reported for a related model [26]. In line with the critical coherence threshold described in this paper, the transition was found to be continuous for two languages and becomes first order when more than two languages are taken into account [14,25].

In this work we follow the view proposed by [14] in considering the investigation of noise induced thresholds for linguistic coherence. In other words, we focus on the study (both for deterministic and stochastic versions) of the effectiveness of the rate of learning errors in precluding the emergence of linguistic consensus. In this framework the coherence threshold is the error rate of language acquisition above which a multilingual community is stable and below which there is a single dominant language.

In the following, we extend the work of [14] by analyzing (i) a family of fitness-driven models that reduce to the Glauber and Voter models in the limit of neutral evolution and (ii) nontrivial networks of interaction. The results detailed below are based on the original Komarova-Niyogi-Nowak (KNN) [14] model with these generalizations, although the time scale for change through learning and selection is no longer generational. As in other models of social interaction and opinion dynamics, learning and selection occur on a shorter time scale, associated with "horizontal" (peer-to-peer) interaction.

The basic assumptions of our family of models are that each individual in the population is a speaker of one of two languages 1 or −1 and that an individual may change its language through interactions with its neighbors. These interactions follow certain rules, where the fitness of the individual and of its neighbors determines the probability for language change in the absence of errors. In line with the usual replicator-mutator dynamics the state update that comes out of these rules is reversed with probability u, which models learning errors as the presence of noise in the system coupled to the dynamics. In order to assess the robustness of the coherence threshold of the KNN model we consider models with more general fitness-driven rules, which reduce to the Voter and Glauber models in the limit of neutral evolution and zero noise. The latter are models of spin dynamics, used to model the mechanisms of opinion dynamics and cultural evolution, which play a role in the evolution of languages on short time scales.

We find that, in general, in well-mixed populations dynamical noise is not effective in driving a critical coherence threshold. In models that reduce to the Voter and Glauber dynamics in the limit of neutral fitness the noise-induced thresholds, separating a dominant language regime from the regime where different languages coexist, become noncritical in the mean-field limit. We derive analytical solutions for the coherence thresholds of the models in complete and regular random graphs that are shown to provide a very good description of the different types of cooperative behavior of this family of models. In particular, the increase in robustness of the coherent linguistic regime as the number of neighbors increases is described quantitatively by the analytical solutions for all dynamical models.

Finally, we put our results in a more general context and provide a complete classification of the threshold behavior of a family of fitness-driven models that includes a flipping rate, or noise uncoupled to the dynamics, instead of the dynamical noise of the replicator-mutator equations.

II. FITNESS-DRIVEN MODELS AND DYNAMICS

We consider the simplest case of the model introduced in [14] characterized by strong selection and two equally fit languages, with no affinity between them. We consider a population of N individuals, where each individual i speaks one of two languages $\sigma_i = \pm 1$, and define the fitness f_i of i as the number of its neighbors that speak the same language:

$$f_i = \sum_j{}' \delta_{\sigma_i,\sigma_j}, \tag{1}$$

where Σ_j' is a sum over the neighbors of i and $\delta_{k,l}$ is 1 if $k=l$ and 0 otherwise. The evolution of the language follows two general rules: (i) the language of the fittest individuals at a given time step (generation) has a higher probability of being learned by the population in the next time step (selection); (ii) in the process of learning there is a probability of error—i.e., a probability that the new generation learns a language with a lower fitness (mutation).

The KNN model of language evolution [14] follows the replicator-mutator rate equations of evolutionary dynamics with fitness functions given by the language frequencies. It was shown in [14] and [16] that, in well-mixed infinite populations, this model exhibits a critical coherence threshold, determined by the rate of learning errors, below which a dominant language is established and maintained in the population. In what follows we analyze the robustness of the linguistic coherence threshold when other mechanisms of evolution and networks of interaction are considered.

In particular, we consider two generalizations of the KNN model by introducing (i) more general fitness-driven dynamics (including additional imitation/social pressure mechanisms) and (ii) populations with nontrivial interaction networks.

We define the social fitness of an individual speaker, $F_{\pm}(i,t)$, as the total fitness of the neighbors of i that speak language ± 1 at generation t,

$$F_{\pm}(i,t) = \sum_j{}' f_j(t)\,\delta_{\sigma_j(t),\pm 1}, \tag{2}$$

and denote by u the probability of learning errors ($0 \leq u \leq 1$). Unless otherwise stated the population has a fixed number N of individuals. The language (of the population) at a given time step t is characterized by the array $\{\sigma_1(t), \sigma_2(t), \dots, \sigma_N(t)\}$. In the next time step the language is determined by the probability that each individual changes its language, through the combined effect of the dynamics and learning errors. We have considered three fitness-driven dynamical models, with learning errors, as detailed below.

A. Replicator-mutator dynamics (KNN model)

In this model the noise, or rate of learning errors, is incorporated in the probability of language change. If $\sigma_i(t-1)=1$, the probability for language change [i.e., the probability that the outcome of the update rule is $\sigma_i(t)=-1$] is given by [14]

$$P_{1\to -1} = \frac{(1-u)F_-(i,t-1) + uF_+(i,t-1)}{F_-(i,t-1) + F_+(i,t-1)}, \qquad (3)$$

while if $\sigma_i(t-1)=-1$ the probability for language change [i.e., the probability that the outcome of the update rule is $\sigma_i(t)=1$] is given by

$$P_{-1\to 1} = \frac{(1-u)F_+(i,t-1) + uF_-(i,t-1)}{F_-(i,t-1) + F_+(i,t-1)}. \qquad (4)$$

As shown in [16], in a well-mixed population the stochastic process with these update rules corresponds in the deterministic limit (infinite population) to the replicator-mutator equations

$$\dot{x}_i = \sum_{j=1}^{2} x_j f_j Q_{ji} - \phi x_i, \qquad (5)$$

where x_i is the frequency of language i and $f_i=x_i$ its fitness, $i=1,2$, $\phi=f_1x_1+f_2x_2$ is the average fitness of the population, and $Q_{11}=Q_{22}=1-u$ and $Q_{12}=Q_{21}=u$ are the elements of the mutation matrix Q.

B. Fitness-driven Voter dynamics

The update rule for this model is inspired in the simplest opinion and imitation dynamics model, the Voter model: a speaker changes language if its fitness is lower than a randomly chosen neighbor that speaks a different language. In addition, with probability u the outcome of the dynamical rule is reversed. The update rule for a speaker i at time t is the following: (i) choose one neighbor j at random; (ii) if $f_j(t-1)>f_i(t-1)$, then $\sigma_i(t)=\sigma_j(t-1)$; (iii) if $f_j(t-1)\le f_i(t-1)$, then $\sigma_i(t)=\sigma_i(t-1)$; (iv) reverse the outcome of the dynamical rule with probability u.

C. Fitness-driven Glauber dynamics

The update rule for this model is inspired in the Glauber dynamics of spin systems, which mimics the effect of social pressure in opinion dynamics. In this model each individual tends to adopt the fittest (dominant) language in its neighborhood. The update rule for a speaker i at time t is the following: (i) if $F_+(i,t-1)>F_-(i,t-1)$, then $\sigma_i(t)=1$; (ii) if $F_+(i,t-1)<F_-(i,t-1)$, then $\sigma_i(t)=-1$; (iii) if $F_+(i,t-1)=F_-(i,t-1)$, then $\sigma_i(t)=\sigma_i(t-1)$; (iv) reverse the outcome of the dynamical rule with probability u.

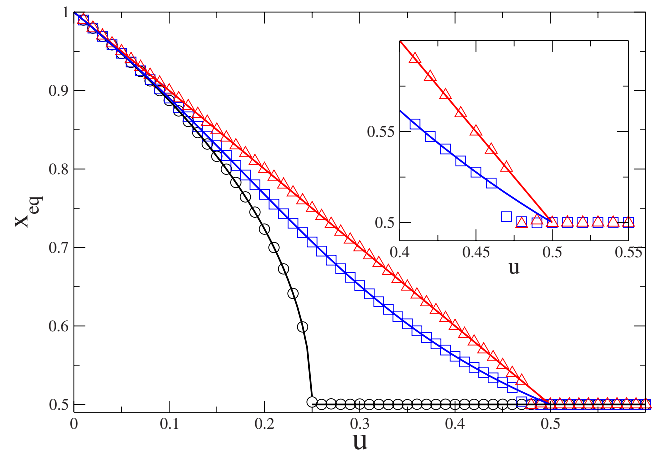

FIG. 1. (Color online) Symbols: mean fraction of speakers of language 1 (x) during 5000 generations for different values of u and a population $N=10^4$ (simulation results in a fully connected network for an initial condition $x=1$). Circles: replicator-mutator dynamics. Triangles: fitness-driven Glauber dynamics. Squares: fitness-driven voter dynamics. Lines: fixed points x^* from (13), (20), and (25) for $x\ge\frac{1}{2}$.

III. SIMULATIONS FOR WELL-MIXED POPULATIONS AND FOR REGULAR RANDOM GRAPHS

The models described above were first simulated on complete graphs or fully connected networks—i.e., where all individuals are neighbors of each other. On these networks the fitness of individuals speaking the same language is identical in each time step (generation): the fitness of individuals speaking +1 (−1) is N_1-1 $(N-N_1-1)$, where N_1 is the total number of speakers of 1.

We start the simulations from a fully ordered system; i.e., at $t=0$ all individuals speak language +1 (say). The language of the next generation is determined according to the rules described above for each model. The language of the population evolves through a large number of generations (5000) and the mean value of $x=\frac{N_1}{N}$ is calculated for each value of error rate in learning, u.

In Fig. 1 we plot the results of simulations of the three dynamics for different population sizes $N=100$, $N=1000$, and $N=10\,000$. Although finite size effects are visible for the smaller systems, they are negligible for populations of thousands. While we find a critical coherence threshold at $u=1/4$, reproducing the results of [14] for the KNN model, the results for the Voter and Glauber fitness-driven models are quite different: the threshold is shifted to the value of the noise that completely overrides the dynamics, $u=1/2$, and the fraction of speakers of the dominant language approaches $x=1/2$ linearly, at the threshold, revealing its noncritical nature.

In order to investigate the effects of the network of interactions on the linguistic coherence threshold we simulated the same models on regular random graphs (RRGs), where

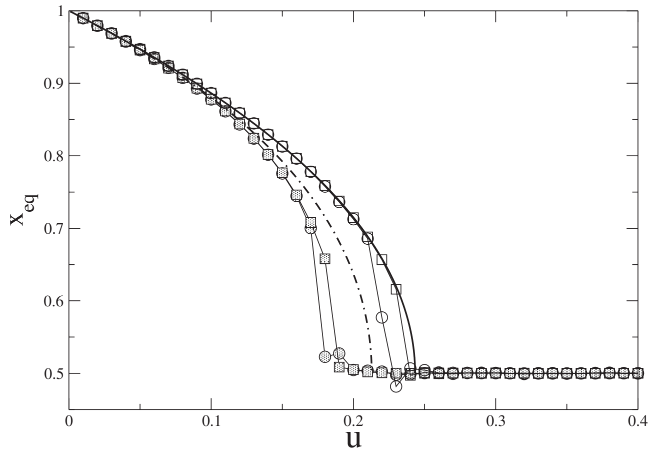

FIG. 2. Symbols: mean fraction of speakers of language 1 as a function of u for the replicator-mutator dynamics from simulations (10 000 generations for an initial condition $x=1$) in a random regular lattice with degree k. Solid symbols: $k=4$. Open symbols: $k=20$. Circles: $N=10^3$. Squares: $N=10^4$. Lines: fixed points x^* from (9) and (16) for $x \geq \frac{1}{2}$. Solid line: $k=20$. Dot-dashed line: $k=4$.

analytical results may also be obtained. In RRG networks N nodes are linked at random to a fixed number of neighbors, k, without double links and self links. The models were simulated on two of these networks for small ($k=4$) and large ($k=20$) degree. The simulations for $N=10^3$ and $N=10^4$ start (as before) in the ordered state where all individuals speak +1. For each value of u, the language evolves through 10 000 generations at the end of which the average fraction of speakers of the dominant language is computed. The results for the three models are plotted in Figs. 2–4.

Note that the transition on RRGs exhibits a critical threshold for all models. The value of u at threshold, u_{th}, increases with the number of neighbors and approaches the MF values ($\frac{1}{4}$ for the replicator mutator and $\frac{1}{2}$ for the Voter and Glauber dynamics) as the number of neighbors tends to infinity. Above threshold $u > u_{th}$, the equilibrium value of x corresponds to the coexistence of the two languages, $x=\frac{1}{2}$.

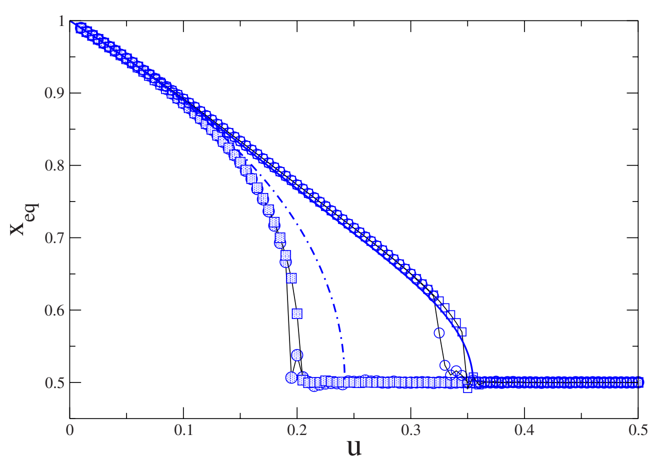

FIG. 3. (Color online) The same as in Fig. 2, but for the fitness-driven Voter dynamics. The fixed points x^* were calculated using (9), (22), and (18) for $x \geq \frac{1}{2}$.

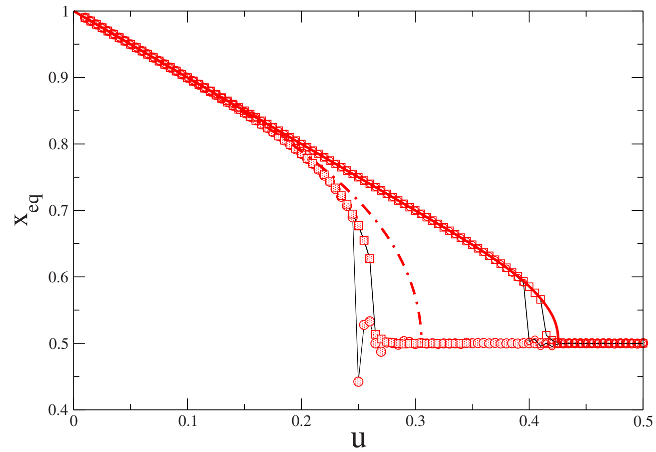

FIG. 4. (Color online) The same as in Fig. 2, but for the fitness-driven Glauber voter dynamics. The fixed points x^* were calculated using (9) and (26) for $x \geq \frac{1}{2}$.

IV. ANALYSIS OF THE MEAN-FIELD EQUATIONS

In order to shed light on these results we proceed to calculate the equilibrium values of N_1 analytically in the infinite population limit. Let x be the fraction of speakers of language 1 ($x \equiv N_1/N$). The evolution of x is given by

$$\dot{x} = -xP_{1\to-1} + (1-x)P_{-1\to1}, \tag{6}$$

where $P_{1\to-1}$ and $P_{-1\to1}$ are the rates of change of the two competing languages.

In well-mixed populations $P_{1\to-1}$ and $P_{-1\to1}$ depend only on x and can be computed exactly for the three models.

On RRG networks these probabilities are calculated using the following (mean-field) assumptions: (i) each of the k neighbors of any site is linked to $(k-1)$ second neighbors, with no loops (uncorrelated links); (ii) the probability that the language spoken at a given site is +1 is the average density of speakers of that language, x (uncorrelated densities). Within this mean-field approximation, the fitness of each node is a random variable that results from the sum of k independent and identical binomial variables. The calculation of the transition probabilities $P_{1\to-1}$ and $P_{-1\to1}$ depends on the specific dynamics and proceeds in a straightforward fashion.

Given the symmetry of the models the probabilities (6) may be written as

$$P_{1\to-1} = (1-u)Q(x) + u[1-Q(x)], \tag{7}$$

$$P_{-1\to1} = (1-u)Q(1-x) + u[1-Q(1-x)], \tag{8}$$

where $Q(x)$ is a function that depends on the network of contacts and on the dynamics. Substituting (7) and (8), Eq. (6) becomes

$$\dot{x} = (1-2u)[-xQ(x) + (1-x)Q(1-x)] + u(1-2x). \tag{9}$$

In what follows we discuss the meaning of $Q(x)$ for each dynamical model and calculate it for each network of contacts.

A. Replicator-mutator dynamics (KNN model)

We find by inspection of (3), (4), (7), and (8) that for the KNN model, $Q(x)$ $[Q(1-x)]$ is the normalized value of F_- $[F_+]$ in the neighborhood of a speaker of language 1 $[-1]$:

$$Q(x) = \frac{F_-}{F_- + F_+}. \tag{10}$$

In well-mixed populations, F_- and F_+ take their mean values and $Q(x)$ becomes

$$Q(x) = \frac{(1-x)^2}{(1-x)^2 + x^2}. \tag{11}$$

Substituting (11) into (9) yields the evolution equation

$$\dot{x} = \frac{[-x(1-x)+u](1-2x)}{x^2 + (1-x)^2}, \tag{12}$$

and the (stable) fixed points x^* are solutions of $\dot{x}=0$. It is straightforward to show that

$$x^* = \begin{cases} \dfrac{1}{2} & \text{if } u > \dfrac{1}{4}, \\ \dfrac{1}{2}(1 \pm \sqrt{1-4u}) & \text{if } u \le \dfrac{1}{4}, \end{cases} \tag{13}$$

confirming that $u_{th}=\frac{1}{4}$ is the threshold for linguistic coherence. Furthermore, this threshold is critical since the derivative of x^* with respect to u diverges there. The function (13) is plotted for $x^* \ge \frac{1}{2}$ in Fig. 1, and excellent agreement is found between the analytical solution and the simulation results for large systems.

The transition probabilities of the KNN model on RRGs are calculated by determining the average value of the total fitnesses $F_+(i)$ and $F_-(i)$ in the neighborhood of a given node. Let us consider a node i that speaks $+1$ with n neighbors that speak also $+1$. The average fitness of one of these neighbors is $1+(k-1)x$ and that of the neighbors speaking -1 is $(k-1)(1-x)$. The average values of $F_+(i)$ and $F_-(i)$ are, then,

$$F_- = (k-n)(k-1)(1-x), \tag{14}$$

$$F_+ = n[1 + (k-1)x]. \tag{15}$$

The number of neighbors of i speaking the same language, n, is a random variable that results from the sum of k random binomial variables, each one taking the value 1 with probability x and 0 with probability $(1-x)$. The function $Q(x)$ in (10) is, then,

$$Q(x) = \sum_{n=0}^{k} B(k,n) x^n (1-x)^{k-n} \frac{F_-}{F_- + F_+}, \tag{16}$$

where F_- and F_+ are given by (14) and (15) and $B(i,j)$ is the binomial coefficient:

$$B(i,j) = \frac{i!}{j!\,(i-j)!}. \tag{17}$$

The evolution equation is obtained by substituting (16) in (9). The stable fixed points as a function of the noise parameter are plotted in Fig. 2 for $k=4$ and $k=20$, respectively. For regular random graphs with $k=20$ the agreement between the simulation and the analytic results is almost quantitative for populations of the order of a few thousand.

B. Fitness-driven Voter dynamics

In the fitness-driven Voter dynamics an individual that speaks language 1 changes to language -1: with probability $(1-u)$ if the neighbor chosen at random speaks -1 and has a higher fitness, with probability u if the neighbor chosen at random speaks 1 or has a lower fitness. Thus, $Q(x)$ $[Q(1-x)]$ is the probability to find a neighbor that speaks -1 $[1]$ and has a higher fitness. In the mean-field approximation, $Q(x)$ is

$$Q(x) = (1-x)H(x), \tag{18}$$

the product of the probability $(1-x)$ of finding a neighbor that speaks -1 and the probability $H(x)$ that a speaker of -1 has higher fitness than a speaker of 1.

In well-mixed populations, the probability that a speaker -1 has a higher fitness is 1 (0) for $x<\frac{1}{2}$ $(x>\frac{1}{2})$, implying that

$$Q(x) = (1-x)\Theta(1-2x), \tag{19}$$

where $\Theta(z)$ is the step function: $\Theta(z)=1$ if $z>0$ and $\Theta(z)=0$ if $z \le 0$. The dynamical equation is obtained by substituting (19) into (9) and has stable fixed point solutions $\dot{x}=0$ given by

$$x^* = \begin{cases} \dfrac{1}{2} & \text{if } u \ge \dfrac{1}{2}, \\ \dfrac{1}{2}[1 \pm (2\alpha - \sqrt{1+4\alpha^2})] & \text{if } u < \dfrac{1}{2}, \end{cases} \tag{20}$$

where $\alpha=\frac{u}{1-2u}$. Again, the rate of learning errors $u^{th}=\frac{1}{2}$ defines two regimes: for $u<u^{th}$ a dominant language is established and maintained while for $u \ge u^{th}$ there is the coexistence of the two equally probable languages. Note, however, that $u^{th}=1/2$ is a trivial threshold in the sense that for this level of noise the evolution is totally random, while for higher levels of noise the evolution rules actually hinder linguistic coherence. This trivial threshold is noncritical since

$$\lim_{u \to 1/2^-} \frac{dx^*}{du} = \pm \frac{1}{2}, \tag{21}$$

the derivative at threshold being finite. The function (20) is plotted for $x^* \ge \frac{1}{2}$ in Fig. 1, and apart from the finite-size effects mentioned previously, quantitative agreement is found between the analytical solution and the simulation results.

To calculate $H(x)$ for RRG, let us consider a node i with $\sigma_i=+1$ and a neighbor j with $\sigma_j=-1$. Using the definition of fitness and the rules of the Voter dynamics we can compute $H(x)$ as the probability that j has a number m of neighbors speaking -1 that is larger than the number n of neighbors of i speaking $+1$:

$$H(x) = \sum_{n=0}^{k-2} B(k-1,n)x^n(1-x)^{k-1-n} \sum_{m=n+1}^{k-1} B(k-1,m) \times(1-x)^m x^{k-1-m}. \quad (22)$$

The stable fixed points are calculated using (9) with $Q(x)$ given by (18) and (22) and solving for $\dot{x}=0$. In general, ($x \neq \frac{1}{2}$) they are more easily written in terms of $u(x^*)$:

$$u(x^*) = \begin{cases} \dfrac{V(x^*)}{2x^* - 1 + V(x^*)} & \text{if } x^* > \dfrac{1}{2}, \\ \dfrac{V(1-x^*)}{1 - 2x^* + V(1-x^*)} & \text{if } x^* < \dfrac{1}{2}, \end{cases} \quad (23)$$

where $V(x)=x(1-x)[H(1-x)-H(x)]$. $u(x^*)$ given by (23) is plotted in Fig. 3 for $x>\frac{1}{2}$. The calculated $x^*(u)$ is in line with the results of the simulation, and for $k=20$ the agreement becomes nearly quantitative.

C. Fitness-driven Glauber dynamics

In the fitness-driven Glauber dynamics, $Q(x)$ is the probability that a speaker i of +1 has total fitness satisfying $F_-(i)>F_+(i)$.

In well-mixed populations, $F_->F_+$ iff $x<\frac{1}{2}$ and, therefore,

$$Q(x) = \Theta(1-2x). \quad (24)$$

Substituting in (9) and solving for the stable fixed points we obtain

$$x^* = \begin{cases} \dfrac{1}{2} & \text{if } u \geq \dfrac{1}{2}, \\ \dfrac{1}{2} \pm \left(\dfrac{1}{2} - u\right) & \text{if } u < \dfrac{1}{2}. \end{cases} \quad (25)$$

The rate of learning errors $u^{th}=\frac{1}{2}$ is again a trivial threshold that separates two regimes as in the fitness-driven Voter model. Also as in the Voter model the threshold is noncritical since, the derivative of u at threshold is finite and the fraction of speakers of the dominant language approaches the threshold linearly. The function (25) is plotted for $x^* \geq \frac{1}{2}$ in Fig. 1, and excellent agreement is found between the analytical solution and the simulation results for large systems.

In order to calculate $Q(x)$ for RRGs, we consider in the neighborhood of a given node i that is a +1 speaker (i) n nearest neighbors that speak +1, (ii) n_1 next-nearest neighbors that speak +1 and share with i a nearest neighbor that speaks +1, and (iii) n_2 next-nearest neighbors that speak −1 and share with i a nearest neighbor that speaks −1. Then the total fitnesses are simply given by $F_-(i)=n_2$ and $F_+(i)=n+n_1$, and $Q(x)$ is the probability that $n_2>n+n_1$:

$$Q(x) = \sum_{n=0}^{k} B(k,n)x^n(1-x)^{k-n} \sum_{n_1=0}^{n(k-1)} B(n(k-1),n_1)x_1^n \times(1-x)^{n(k-1)-n_1} \sum_{n_2=n+n_1+1}^{(k-n)(k-1)} B((k-n)(k-1),n_2) \times(1-x)^{n_2}x^{(k-n)(k-1)-n_2}. \quad (26)$$

The fixed points are calculated using (9) and solving for $\dot{x}=0$. The function $u(x^*)$ obtained is of the form (23) with $V(x)=-xQ(x)+(1-x)Q(1-x)$ and $Q(x)$ given by (26). $u(x^*)$ is plotted in Fig. 4 for $x>\frac{1}{2}$ and is found to be in line with the results of the simulations. For $k=20$ the agreement is almost quantitative.

Again, the transition exhibits a critical threshold and the value of u_{th} also increases with the number of nearest neighbors on the network approaching the mean-field value, $u_{th}=1/2$, as this number approaches infinity. Above threshold, $u>u_{th}$, the stable fixed point corresponds to the coexistence of the two languages, $x=\frac{1}{2}$.

D. Coherence thresholds for social fitness-driven evolution

For all the models considered above, the mean-field description of the dynamics is given by Eq. (9) which is of the form

$$\dot{x} = (1-2u)g(x) + u(1-2x), \quad (27)$$

with $g(1/2)=0$ and $g(x)=-g(1-x)$. For (27) to describe the mean-field dynamics of an evolution process that selects for the dominant variant of two languages or species with the same intrinsic fitness, the additional assumptions are that $g(0)=0$ and $g(x)<0$ for $0<x<1/2$, so that the coherent states $x=0$ and $x=1$ are the only stable solutions in the absence of noise. The threshold behavior of this type of models is easily understood if we consider the related family

$$\dot{x} = \tilde{g}(x) + u(1-2x), \quad (28)$$

where $\tilde{g}(x)$ has the same symmetry properties as $g(x)$ and noise and dynamics are uncoupled, so that u represents a constant rate of random flipping independent of the evolution rules. Indeed, (27) can be brought to the form (28) with $\tilde{g}(x)=g(x)(1-2x)/[1-2x-2g(x)]$ through a smooth rescaling of time, provided that $g(x)$ is smooth. From Eq. (28), the curve $u(x)$ that relates the rate of random flipping u with the corresponding equilibrium density x is

$$u(x) = \frac{\tilde{g}(x)}{2x-1}. \quad (29)$$

If we assume that $\tilde{g}(x)$ is smooth, then given the symmetry

$$\tilde{g}(x) = \tilde{g}'(1/2)(x-1/2) + O((x-1/2)^3) \quad (30)$$

and therefore

$$u^{th} = u(1/2) = \tilde{g}'(1/2)/2, \quad \frac{du^{th}}{dx} = u'(1/2) = 0. \quad (31)$$

This means that models (28) and (27) always exhibit critical coherence thresholds when $\tilde{g}$ and g are smooth. The values of the critical thresholds found in this section are particular cases of Eq. (31), which for model (27) reads

$$u^{th} = u(1/2) = \frac{1}{2}\frac{g'(1/2)}{1+g'(1/2)}, \quad \frac{du^{th}}{dx} = u'(1/2) = 0. \quad (32)$$

Equation (29) also shows that whenever $\tilde{g}(x)$ is a step function with a discontinuity at $x=1/2$, then model (28) has no coherence threshold: a dominant language persists for arbitrarily large levels of noise, because the right and left limits $u(1/2^{+})$ and $u(1/2^{-})$ are both infinite. The behavior of model (27) when $g(x)$ is a step function with a discontinuity at $x=1/2$, as for the fitness-driven Voter and Glauber dynamics on the complete graph, may be obtained directly from the analog of Eq. (29) for model (27):

$$u(x) = \frac{g(x)}{2x - 1 + 2g(x)}. \tag{33}$$

Then

$$u^{th} = u(1/2^{+}) = u(1/2^{-}) = 1/2, \tag{34}$$

independently of g, and

$$\frac{du^{th}}{dx} = u'(1/2^{+,-}) = -\frac{1}{2}\frac{1}{g(1/2^{+,-})}, \tag{35}$$

which is always bounded away from zero. Therefore, this class of models will exhibit a noncritical trivial threshold at $u=1/2$.

To summarize, the family of models described at the mean-field level by Eqs. (27) and (28) exhibits three types of threshold behavior. Models with nonsmooth density dependence transition rates $g(x)$ and dynamically coupled noise exhibit a trivial noncritical threshold at the value of $u=1/2$ for which noise completely overrides the dynamics. Models with nonsmooth density dependence transition rates $\tilde{g}(x)$ and dynamically uncoupled noise do not exhibit a noise-induced threshold; i.e., there is always a dominant language irrespective of the level of noise. Finally, in the generic case of models with smooth transition rates $g(x)$ and $\tilde{g}(x)$ there is a critical threshold below which language coherence may be established and maintained.

V. CONCLUSIONS

We have considered different models for the evolution in the presence of noise of two languages with the same intrinsic fitness that compete through the selective advantage of the language that is perceived by each individual as the dominant language. The language spoken by each speaker has for that speaker a social fitness given by the number of its neighbors that share the same language, and the dynamics-driven by evolution rules based on this fitness measure will depend also on the interaction network of the population. Starting from a state where all individuals speak the same language, mutations or transmission errors act as noise terms that favor the balance of the number of speakers of each language, while selection according to social fitness drives linguistic coherence. The coherence threshold is the level of noise or mutation rate above which the system evolves to a state where both languages are equally frequent.

From simulations of these models on fully connected networks and on regular random graphs, we found that the critical threshold for the KNN model [14] is robust with respect to the network structure, but that the coherence thresholds of related models are strongly affected by this feature.

In particular, we have found that models with social fitness-driven dynamics inspired by the Voter and Glauber models, two of the simplest models for spin dynamics used in opinion dynamics and cultural evolution studies, exhibit different linguistic coherence threshold behavior, depending on the network of interactions. On a regular random graph, these models have a critical coherence threshold, while on the fully connected network a dominant language persists up to the level of noise for which the evolution rules are totally random.

We have obtained analytical mean-field solutions for the coherence thresholds on the fully connected network and on regular random networks that are in agreement with the results of the simulations for the three models, providing a quantitative description of the behavior of the different microscopic rules. We have shown that the noise threshold behaviors of these models, and, more generally, of evolution processes that select for the dominant variant of two languages or species with the same intrinsic fitness, can be understood as well in terms of a simple mean-field analysis.

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